

Bounded operators on the weighted spaces of holomorphic functions on the unit Ball in C^n

A. V. Harutyunyan¹ and W.Lusky²

¹Department of Applied Mathematics, Yerevan State University
1 Alex Manookian str., 0025, Yerevan, Armenia
anahit@ysu.am

²Institute of Mathematics, University of Paderborn
100 Warburger str., 33098, Paderborn, Germany
lusky@math.uni-paderborn.de

July 2, 2014

Abstract

Assuming that S is the space of functions of regular variation, $\omega \in S$, $0 < p < \infty$, a function f holomorphic in B^n is said to be of Besov space $B_p(\omega)$ if

$$\|f\|_{B_p(\omega)}^p = \int_{B^n} (1 - |z|^2)^p |Df(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1}} d\nu(z) < +\infty,$$

where $d\nu(z)$ is the volume measure on B^n and D stands for a fractional derivative of f .

We consider operators on $B_p(\omega)$ and show, that they are bounded.

AMS Subject Classification: 32C37, 47B38, 46T25, 46E15.

Key Words and Phrases: Weighted Besov spaces, Unit ball, Operator.

1 Introduction and basic constructions

Let C^n denote the complex Euclidean space of a dimension n . For any points $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$ in C^n , we define the inner product as $\langle z, \zeta \rangle = z_1 \bar{\zeta}_1 + \dots + z_n \bar{\zeta}_n$ and note that $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. By $B^n = \{z \in C^n, |z| < 1\}$ and $C^n : S^n = \{z \in C^n, |z| = 1\}$ we denote the open unit ball and its boundary, i.e. the unit sphere, in C^n . Further, by $H(B^n)$ we denote the set of holomorphic functions on B^n and by $H^\infty(B^n)$ the set of bounded holomorphic functions on B^n .

If $f \in H(B^n)$, then $f(z) = \sum_m a_m z^m$ ($z \in B^n$), where the sum is taken over all multiindices $m = (m_1, \dots, m_n)$ with nonnegative integer components m_k and $z^m = z_1^{m_1} \dots z_n^{m_n}$. Assuming that $|m| = m_1 + \dots + m_n$ and putting $f_k(z) = \sum_{|m|=k} a_m z^m$ for any $k \geq 0$, one can rewrite the Taylor expansion of f as

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \quad (1)$$

which is called homogeneous expansion of f , since each f_k is a homogeneous polynomial of the degree k . Further, for a holomorphic function f the fractional differential D^α is defined as

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(z),$$

$$D^\alpha f(\bar{z}) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(\bar{z}), \quad k = (k_1, \dots, k_n), \quad z \in B^n.$$

We consider the inverse operator $D^{-\alpha}$ defined in the standard way:

$$D^{-\alpha} D^\alpha f(z) = f(z).$$

Particularly, $D^1 f(z) = Df(z)$ if $\alpha = 1$.

The following properties of D are evident

1. $DD^\alpha f(z) = D^{\alpha+1} f(z)$
2. $D^m (1 - \langle z, \zeta \rangle)^{-\alpha} \preceq (1 - \langle z, \zeta \rangle)^{-\alpha-m}$

By $d\nu$ we denote the volume measure on B^n , normalized so that $\nu(B^n) = 1$, and by $d\sigma$ the surface measure on S^n , normalized so that $\sigma(S^n) = 1$. Then following lemma, the proof of which can be found in [7] or [12], reveals the connection between these measures.

Lemma 1. *If f is a measurable function with summable modulus over B^n , then*

$$\int_{B^n} f(z) d\nu(z) = 2n \int_0^1 r^{2n-1} dr \int_{S^n} f(r\zeta) d\sigma(\zeta).$$

Definition 1. By S we denote the well-known class of all non-negative measurable functions ω on $(0, 1)$ with

$$\omega(x) = \exp \left\{ \int_x^1 \frac{\varepsilon(u)}{u} du \right\}, \quad x \in (0, 1),$$

where $\varepsilon(u)$ is some measurable, bounded functions on $(0, 1)$ and $-\alpha_\omega \leq \varepsilon(u) \leq \beta_\omega$.

Note that the functions of S are called *functions of regular variation* (see [10]). Throughout the paper, we shall assume that $\omega \in S$. Throughout the paper the capital Letters $C(\dots)$ and C_k stand for different positive constants depending only on the parameters indicated.

We define the holomorphic Besov spaces on the unit ball as follows(see [3]).

Definition 2. Let $\omega \in S$, $0 < p < \infty$. A function $f \in H(B^n)$ is said to be of $B_p(\omega)$ if

$$M_f^p(\omega) = \int_{B^n} (1 - |z|^2)^p |Df(z)|^p \frac{\omega(1 - |z|)}{(1 - |z|^2)^{n+1}} d\nu(z) < +\infty.$$

We introduce the norm in $H(B^n)$ as $\|f\|_{B_p(\omega)} = M_f(\omega)$ ($|f(0)|$ is not to be added since $Df = 0$ implies $f = 0$ for a holomorphic function f). Besides, it is easy to check that if $p > 1$, $n = 1$ and $\omega(t) = 1$, then $B_p(\omega)$ becomes the classical Besov space (see [1], [5], [11]).

In particular, for $p = +\infty$ we shall write $B_\infty(\omega) = B_\omega$, where B_ω denotes the ω -weighted Bloch space on the ball (see [4]).

In [6], [8], [9], one can see some other definitions and some characterizations of holomorphic Besov spaces on B^n .

Let $1 \leq p < \infty$ and let $f \in B_p(\omega)$. Further, let $m > -n/p - \beta_\omega/p$. Then the function $Df(z)$ has the representation

$$Df(z) = C(\pi, m) \int_{B^n} \frac{(1 - |\zeta|^2)^m Df(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+1+m}} d\nu(\zeta), \quad z \in B^n, \quad (2)$$

where $C(n, m) = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)}$, follows as a simple consequence of that well known in the one-dimensional case (for details, see [2], [12]).

The following auxiliary lemma will be used.

Lemma 2. If $1 \geq p < \infty$ and $f \in B_p(\omega)$, then

$$|f(z)| \leq C(\pi, m) \int_{B^n} \frac{(1 - |\zeta|^2)^m}{|1 - \langle z, \zeta \rangle|^{n+m}} |Df(\zeta)| d\nu(\zeta)$$

for $m \in \mathbb{N}$ and $m > -n/p - \beta_\omega/p$.

Proof. Obviously, $f(z) = \int_0^1 Df(rz)dr$, and by (2) we get

$$\begin{aligned} f(z) &= C(\pi, m) \int_0^1 \int_{B^n} \frac{(1 - |\zeta|^2)^m Df(\zeta)}{(1 - r \langle z, \zeta \rangle)^{n+1+m}} d\nu(\zeta) dr \\ &= C(\pi, m) \int_{B^n} (1 - |\zeta|^2)^m Df(\zeta) \int_0^1 \frac{dr}{(1 - r \langle z, \zeta \rangle)^{n+1+m}} d\nu(\zeta) \\ &= \tilde{C}(\pi, m) \int_{B^n} \frac{(1 - |\zeta|^2)^m ((1 - \langle z, \zeta \rangle)^{n+\alpha} - 1)}{\langle z, \zeta \rangle (1 - \langle z, \zeta \rangle)^{n+m}} Df(\zeta) d\nu(\zeta). \end{aligned}$$

It is clear that $((1 - \langle z, \xi \rangle)^{n+m+1} - 1) / \langle z, \xi \rangle$ is bounded in B^n . Hence the desired statement follows. \square

Lemma 3. Let $\omega \in S$ and let $f \in B_p(\omega)$ for some $0 < p \leq 1$. Then

$$\left(\int_{B^n} |Df(z)| \frac{\omega^{1/p}(1 - |z|)}{(1 - |z|)^n} d\nu(z) \right)^p \leq \int_{B^n} |Df(z)|^p \frac{(1 - |z|)^p \omega(1 - |z|)}{(1 - |z|)^{n+1}} d\nu(z)$$

Proof. We have $|Df(z)| = |Df(z)|^p |Df(z)|^{1-p}$. By Lemma 2 we get

$$|Df(z)| \leq |Df(z)|^p \frac{\|f\|_{B^p(\omega)}^{1-p}}{\omega^{(1-p)/p}(1 - |z|)(1 - |z|)^{1-p}}.$$

Therefore

$$|Df(z)| \frac{(1 - |z|)\omega^{1/p}(1 - |z|)}{(1 - |z|)^{n+1}} \leq |Df(z)|^p \|f\|_{B^p(\omega)}^{1-p} \frac{\omega(1 - |z|)(1 - |z|)^p}{(1 - |z|)^{n+1}},$$

and by integration over B^n we get

$$\int_{B^n} |Df(z)| \frac{(1 - |z|)\omega^{1/p}(1 - |z|)}{(1 - |z|)^{n+1}} d\nu(z) \leq \|f\|_{B^p(\omega)}^{1-p} \int_{B^n} |Df(z)|^p \frac{\omega(1 - |z|)(1 - |z|)^p}{(1 - |z|)^{n+1}} d\nu(z),$$

The proof is completed. \square

Lemma 4. Let $\omega \in S$, $\alpha + 1 - \beta_\omega > 0$, and $\beta - \alpha > \alpha_\omega$. Then

$$\int_{B^n} \frac{(1 - |\zeta|^2)^\alpha \omega(1 - |\zeta|)}{|1 - \langle z, w \rangle|^{\beta+n+1}} d\nu(\zeta) \leq C(\alpha, \beta, \omega) \frac{\omega(1 - |z|^2)}{(1 - |z|^2)^{\beta-\alpha}}.$$

Proof. By Lemma 1 for $\beta > 0$ we get

$$\int_{B^n} \frac{(1 - |\zeta|^2)^\alpha \omega(1 - |\zeta|)}{|1 - \langle z, \zeta \rangle|^{\beta+n+1}} d\nu(\zeta) = 2n \int_0^1 r^{2n-1} (1 - r^2)^\alpha \omega(1 - r) dr \times$$

$$\int_{S^n} \frac{d\sigma(\zeta)}{|1 - \langle z, \zeta \rangle|^{\beta+n+1}} \leq 2n \int_0^1 r^{2n-1} \frac{(1-r^2)^\alpha \omega(1-r)}{(1-r|z|)^{\beta+1}} dr.$$

In the last inequality we have used Theorem 1. 12 from [12].

The problem is to estimate the last one dimensional integral. To this end we have

$$\begin{aligned} & \int_0^1 \frac{(1-r^2)^\alpha \omega(1-r)}{(1-r|z|)^{\beta+1}} dr \leq \int_0^1 \frac{u^\alpha \omega(u) du}{(1-|z|+u|z|)^{\beta+1}} \\ &= \left\{ \int_0^{1-|z|} \frac{u^\alpha \omega(u) du}{(1-|z|+u|z|)^{\beta+1}} + \int_{1-|z|}^1 \frac{u^\alpha \omega(u) du}{(1-|z|+u|z|)^{\beta+1}} \right\} = I_1 + I_2. \end{aligned}$$

First we estimate the integral I_1 .

$$\begin{aligned} I_1 &\leq \int_0^{1-|z|} \frac{u^\alpha \omega(u)}{(1-|z|)^{\beta+1}} du = \frac{1}{(1-|z|)^{\beta+1}} \int_0^{1-|z|} u^\alpha \omega(u) du = \\ & \frac{(\alpha+1)^{-1}}{(1-|z|)^{\beta+1}} \left[(1-|z|)^{\alpha+1} \omega(1-|z|) + \int_0^{1-|z|} u^\alpha \omega(u) \varepsilon(u) du \right] \end{aligned}$$

As a result we get

$$(\alpha+1) \int_0^{1-|z|} u^\alpha \omega(u) du = (1-|z|)^{\alpha+1} \omega(1-|z|) + \int_0^{1-|z|} u^\alpha \omega(u) \varepsilon(u) du,$$

and

$$\int_0^{1-|z|} (\alpha+1-\varepsilon(u)) u^\alpha \omega(u) du = (1-|z|)^{\alpha+1} \omega(1-|z|).$$

On the other hand

$$\alpha+1-\beta_\omega \leq \alpha+1-\varepsilon(u)$$

which yields

$$(\alpha+1-\beta_\omega) \int_0^{1-|z|} u^\alpha \omega(u) du \leq (1-|z|)^{\alpha+1} \omega(1-|z|)$$

or

$$I_1 \leq C(\alpha, \beta, \omega) \frac{\omega(1-|z|)}{(1-|z|)^{\beta-\alpha}} \quad (3)$$

Now we want to estimate I_2 . For $|z| \geq 1/2$ we have

$$I_2 \leq 2^{\beta-1} \int_{1-|z|}^1 \frac{\omega(u)}{u^{\beta-\alpha+1}} du = \frac{2^\beta}{\alpha-\beta-1} \int_{1-|z|}^1 \omega(u) du^{\alpha-\beta-1}$$

$$= \frac{2^{\beta-1}}{\beta - \alpha + 1} \left[\frac{\omega(1 - |z|)}{(1 - |z|)^{\beta-\alpha-1}} + \omega(1) + \int_{1-|z|}^1 \frac{\omega(u)\varepsilon(u)}{u^{\beta-\alpha}} du \right]$$

Then

$$\int_{1-|z|}^1 \frac{\omega(u)}{u^{\beta-\alpha}} du = \frac{\omega(1 - |z|)}{(1 - |z|)^{\beta-\alpha-1}} - \omega(1) - \int_{1-|z|}^1 \frac{\omega(u)\varepsilon(u)}{u^{\beta-\alpha}} du$$

and it follows that

$$\int_{1-|z|}^1 \left(1 + \frac{\varepsilon(u)}{\beta - \alpha - 1} \right) \frac{\omega(u)}{u^{\beta-\alpha}} du = \frac{\omega(1 - |z|)}{(1 - |z|)^{\beta-\alpha-1}} - \omega(1) \leq \frac{\omega(1 - |z|)}{(1 - |z|)^{\beta-\alpha-1}}.$$

Then the inequality

$$1 + \frac{\varepsilon(u)}{\beta - \alpha - 1} \geq 1 - \frac{\alpha_\omega}{\beta - \alpha - 1} > 0,$$

gives us

$$\int_{1-|z|}^1 \frac{\omega(u)}{u^{\beta-\alpha}} du \leq C(\alpha, \beta, \omega) \frac{\omega(1 - |z|)}{(1 - |z|)^{\beta-\alpha-1}}. \quad (4)$$

Summing up, from (3) and (4) we get the proof of Lemma 4. \square

Lemma 5. *The following statement is true*

$$D(fg) = gDf + fDg - fg$$

.

Proof. To this end we define the radial derivative R of f as follows

$$(Rf)(z) = \sum_{k=1}^{\infty} k f_k(z), \quad z \in B^n$$

or equivalently

$$(Rf)(z) = \sum_{k=1}^n z_k \frac{\partial f(z)}{\partial z_k}, \quad z \in B^n.$$

It is easy to note that $R(fg) = gR(f) + fR(g)$ and for the operator D we have $Df = f + Rf$. Combining the last equalities we get

$$D(fg) = fg + gR(f) + fR(g).$$

On the other hand we have $R(f) = Df - f$ and $R(g) = Dg - g$. Hence $D(fg) = gDf + fDg - fg$. \square

2 Bounded Operators on $B_p(\omega)$

In this Section first we consider the following operator

$$T_h^\alpha(f)(z) = \int_{B^n} \frac{(1 - |\xi|^2)^\alpha \overline{h(\xi)} f(\xi)}{(1 - \langle z, \xi \rangle)^{n+\alpha+1}} d\vartheta(\xi), \quad \alpha > -1.$$

Theorem 1. *Let $0 < p < \infty$, $h \in H^1(B^n)$. Then*

1. *if T_h^α is bounded on $B_p(\omega)$ then $h \in H^\infty(B^n)$.*
2. *conversely,*
 - a) *if $1 \leq p < \infty$ and $h \in H^\infty(B^n)$ then $T_h^\alpha : B_p(\omega) \rightarrow B_p(\omega)$*
 - b) *if $0 < p < 1$ and $h \in H^\infty(B^n)$ then $T_h^\alpha : B_p(\omega) \rightarrow B_p(\omega^*)$, where $\omega^*(t) = t^{(\alpha+m+1)(1-p)}\omega(t)$ and $m > -n/p - \beta_\omega/p$.*

Proof. 1. Let T_h^α be bounded on $B_p(\omega)$. We take

$$f_\tau(\xi) = \frac{1}{(1 - \langle \xi, \tau \rangle)^{\alpha+n+1}},$$

where $\tau \in [0, 1]^n$ is a parameter. We calculate

$$\begin{aligned} T_h^\alpha(f)(z) &= \int_{B^n} \frac{(1 - |\xi|^2)^\alpha \overline{h(\xi)} d\vartheta(\xi)}{(1 - \langle z, \xi \rangle)^{n+\alpha+1} (1 - \langle \xi, \tau \rangle)^{n+\alpha+1}} \\ &= \int_{B^n} \frac{(1 - |\xi|^2)^\alpha h(\xi) d\vartheta(\xi)}{(1 - \langle \xi, z \rangle)^{n+\alpha+1} (1 - \langle \tau, \xi \rangle)^{n+\alpha+1}} = \frac{\overline{h(z)}}{(1 - \langle z, \tau \rangle)^{n+\alpha+1}} \end{aligned}$$

Then we get

$$\|T_h(f_\tau)\|_{B_p(\omega)} = |h(z)| \cdot \|f_\tau\|_{B_p(\omega)} \leq \|T_h\| \cdot \|f\|_{B_p(\omega)}$$

and hence $|h(\tau)| \leq \|T_h\|$. Replacing $f_\tau(z)$ by $f_\tau(e^{i\theta}z)$ we get $|h(\tau e^{i\theta})| \leq \|T_h\|$ which implies $h \in H^\infty(B^n)$.

2. Conversely, a) let $p \geq 1$ and $h \in H^\infty(B^n)$. We show that $T_h^\alpha(f) \in B_p(\omega)$ for any $f \in B_p(\omega)$. To this end by Lemma 2 we use the inequality

$$|f(\xi)| \leq C(\pi, m) \int_{B^n} \frac{(1 - |t|^2)^m |Df(t)|}{|1 - \langle \xi, t \rangle|^{m+n}} d\vartheta(t)$$

which implies that

$$|f(\xi)|^p \leq \frac{C(\pi, m)}{(1 - |\xi|^2)^{(m-1)p/q}} \int_{B^n} \frac{(1 - |t|^2)^{mp} |Df(t)|^p}{|1 - \langle \xi, t \rangle|^{m+n}} d\vartheta(t)$$

Then for $p > 1$, by Holders inequality, we get,

$$\begin{aligned} |T_h^\alpha f(z)|^p &\leq C(\pi, m) \left(\int_{B^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)| \cdot |\overline{h(\xi)}| d\vartheta(\xi)}{|1 - \langle z, \xi \rangle|^{n+2+\alpha}} \right)^p \\ &\leq C(\pi, m) \frac{\|h\|_\infty}{(1 - |z|^2)^{p/q}} \int_{B^n} \frac{(1 - |\xi|^2)^\alpha |f(\xi)|^p d\vartheta(\xi)}{|1 - \langle z, \xi \rangle|^{n+2+\alpha}}. \end{aligned}$$

Then we have

$$\begin{aligned} I &\equiv C(\pi, m) \int_{B^n} |DT_h^\alpha f(z)|^p \frac{\omega(1 - |z|) d\vartheta(z)}{(1 - |z|^2)^{n+1-p}} \\ &\leq C(\pi, m) \int_{B^n} (1 - |t|^2)^{mp} |Df(t)|^p \int_{B^n} \frac{(1 - |\xi|^2)^{\alpha - (m-1)p/q}}{|1 - \langle \xi, t \rangle|^{m+n}} \\ &\quad \int_{B^n} \frac{\omega(1 - |z|) d\vartheta(z) d\vartheta(\xi) d\vartheta(t)}{(1 - |z|^2)^{n+1-p+p/q} |1 - \langle z, \xi \rangle|^{n+2+\alpha}} \end{aligned}$$

Using Lemma 4 we obtain furthermore

$$\begin{aligned} I &\leq C(\pi, m) \int_{B^n} (1 - |t|^2)^{mp} |Df(t)|^p \int_{B^n} \frac{(1 - |\xi|^2)^\alpha \omega(1 - |\xi|) d\vartheta(z) d\vartheta(\xi)}{|1 - |\xi|^2|^{n+1+\alpha+(m-1)p/q} |1 - \langle \xi, t \rangle|^{n+m}} \\ &\leq C(\pi, m) \|h\|_\infty \int_{B^n} \frac{(1 - |t|^2)^{mp} |Df(t)|^p \omega(1 - |t|) d\vartheta(t)}{(1 - |t|^2)^{(m-1)p/q + m+n}} \\ &= C(\pi, m) \|h\|_\infty \int_{B^n} \frac{|Df(\xi)|^p \omega(1 - |\xi|) d\vartheta(\xi)}{|1 - |\xi|^2|^{n+1-p}} \leq C(\pi, m) \|f\|_{B_p(\omega)}^p \|h\|_\infty. \end{aligned}$$

Let now $p = 1$. We have

$$\begin{aligned}
& \int_{B^n} |DT_h^\alpha f(z)| \frac{\omega(1-|z|)d\vartheta(z)}{(1-|z|^2)^n} \\
& \leq C(\pi, m) \|h\|_\infty \int_{B^n} \frac{(1-|w|^2)^m |Df(w)|}{|1-\langle \xi, w \rangle|^{m+n}} \int_{B^n} \frac{(1-|\xi|^2)^\alpha}{|1-\langle z, w \rangle|^{n+\alpha+2}} \\
& \quad \int_{B^n} \frac{\omega(1-|\xi|)d\vartheta(z)d\vartheta(\xi)d\vartheta(w)}{|1-|\xi|^2|^n} \\
& = C(\pi, m) \|h\|_\infty \int_{B^n} (1-|w|^2)^m |Df(w)| \int_{B^n} \frac{(1-|\xi|^2)^\alpha}{|1-\langle \xi, w \rangle|^{n+m}} \\
& \quad \int_{B^n} \frac{\omega(1-|z|)d\vartheta(z)d\vartheta(\xi)d\vartheta(w)}{(1-|z|^2)^n |1-\langle z, \xi \rangle|^{n+2+\alpha}} \\
& C(\pi, m) \leq \|h\|_\infty \int_{B^n} (1-|w|^2)^m |Df(w)| \int_{B^n} \frac{(1-|\xi|^2)^\alpha \omega(1-|\xi|)d\vartheta(\xi)d\vartheta(w)}{|1-\langle \xi, w \rangle|^{m+n} (1-|\xi|^2)^{n+1+\alpha}} \\
& C(\pi, m) \leq \|h\|_\infty \int_{B^n} (1-|w|^2)^m |Df(w)| \frac{\omega(1-|w|)(1-|w|^2)^\alpha d\vartheta(w)}{|1-|w|^2|^{n+m+\alpha}} \\
& = \int_{B^n} \frac{|Df(w)|\omega(1-|w|)}{(1-|w|^2)^n} d\vartheta(w) = C(\pi, m) \|f\|_{B_p(\omega)} \|h\|_\infty.
\end{aligned}$$

b) Let $0 < p < 1$. Using Lemma 3 we get

$$|f(\xi)|^p \leq C(\pi, m) \int_{B^n} \frac{(1-|t|^2)^{mp+p} |Df(t)|^p}{|1-\langle \xi, t \rangle|^{m+n+1}} d\vartheta(t)$$

and

$$|DT_h^\alpha f(z)|^p \leq C(\pi, m) \|h\|_\infty^p \int_{B^n} \frac{(1-|\xi|^2)^{p(\alpha+1)} |f(\xi)|^p d\vartheta(\xi)}{|1-\langle z, \xi \rangle|^{n+2+\alpha}}.$$

As in the case of $p > 1$ by Lemma 4 we get

$$\begin{aligned}
& \int_{B^n} |DT_h^\alpha f(z)|^p \frac{\omega^*(1-|z|)d\vartheta(z)}{(1-|z|^2)^{n+1-p}} \\
& C(\pi, m) \leq \int_{B^n} (1-|t|^2)^{mp+p} |Df(t)|^p \int_{B^n} \frac{(1-|\xi|^2)^{p(\alpha+1)}}{|1-\langle \xi, t \rangle|^{m+n+1}} \\
& \quad \int_{B^n} \frac{\omega^*(1-|z|)d\vartheta(z)d\vartheta(\xi)d\vartheta(t)}{(1-|z|^2)^{n+1-p}|1-\langle z, \xi \rangle|^{n+2+\alpha}} \\
& C(\pi, m) \leq \int_{B^n} (1-|t|^2)^{mp+p} |Df(t)|^p \int_{B^n} \frac{(1-|\xi|^2)^{p(\alpha+1)}\omega^*(1-|\xi|)d\vartheta(z)d\vartheta(\xi)}{(1-|\xi|^2)^{n+1-p+\alpha+2}(1-\langle \xi, t \rangle)^{n+m+1}} \\
& C(\pi, m) \leq \|h\|_\infty \int_{B^n} \frac{(1-|t|^2)^{mp+p} |Df(t)|^p \omega^*(1-|t|)d\vartheta(t)}{(1-|t|^2)^{m+n+2-2p-p\alpha+\alpha}} \\
& = \|h\|_\infty \int_{B^n} \frac{|Df(\xi)|^p \omega(1-|\xi|)d\vartheta(\xi)}{|1-|\xi|^2|^{n+1-p}} \leq C(\pi, m) \|f\|_{B_p(\omega)}^p \|h\|_\infty.
\end{aligned}$$

□ The next theorem is about boundedness M_h

Theorem 2. *Let $H^\infty(B^n)$. Then M_h is a bounded operator $B_p(\omega) \rightarrow B_p(\omega)$.*

Proof. Using Lemma 5 we show that

$$\int_{B^n} |Df(z)|^p |g(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1-p}} d\vartheta(z) < \infty \quad (3)$$

$$\int_{B^n} |f(z)|^p |Dg(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1-p}} d\vartheta(z) < \infty \quad (4)$$

$$\int_{B^n} |f(z)|^p |g(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1-p}} d\vartheta(z) < \infty \quad (5)$$

The proof of (3) is evident.

Proof of (4). First we show that

$$|Dg(z)| \leq \frac{C\|g\|_\infty}{1-|z|^2}, \quad z \in B^n,$$

if $g \in H^\infty(B^n)$. To this end we take the ball $B^n(z) = \{w \in B^n, |w-z| < n-|z|/2\}$ and use the Cauchy inequality. In the case $p > 1$ we have

$$\begin{aligned}
|f(z)|^p & \leq \left(C(\pi, m) \int_{B^n} \frac{(1-|\xi|^2)^m |Df(\xi)| d\vartheta(\xi)}{|1-\langle m, \xi \rangle|^{m+n}} \right)^p \\
& \leq \frac{C(\pi, m)}{(1-|z|^2)^{\gamma p/q}} \int_{B^n} \frac{(1-|\xi|^2)^{m-1-\gamma} (1-|\xi|^2)^{p+\gamma p} |Df(\xi)|^p d\vartheta(\xi)}{|1-\langle z, \xi \rangle|^{m+n}}
\end{aligned}$$

Then for (4) we get

$$\begin{aligned}
& \int_{B^n} |f(z)|^p |Dg(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1-p}} d\vartheta(z) \\
& \leq C(\pi, m) \|g\|_\infty^p \int_{B^n} (1-|\xi|^2)^{m+(p-1)(\gamma+1)} |Df(\xi)|^p \int_{B^n} \frac{\omega(1-|z|^2) d\vartheta(z) d\vartheta(\xi)}{|1-\langle z, \xi \rangle|^{m+n} (1-|z|^2)^{n+2-p+\gamma p/q}} \\
& \leq C(\pi, m) \|g\|_\infty \int_{B^n} (1-|\xi|^2)^{m+(p-1)(\gamma+1)} \frac{|Df(\xi)|^p \omega(1-|\xi|)}{(1-|\xi|)^{m+n-p+\gamma p/q}} d\vartheta(\xi) \\
& = C(\pi, m) \|g\|_\infty^p \int_{B^n} \frac{|Df(\xi)|^p \omega(1-|\xi|) d\vartheta(\xi)}{(1-|\xi|)^{n+1-2p}} \\
& = \int_{B^n} \frac{|Df(\xi)|^p \omega(1-|\xi|)(1-|\xi|)^p d\vartheta(\xi)}{(1-|\xi|^2)^{n+1-p}} \leq \|f\|_{B_p(\omega)}^p C(\pi, m) \|g\|_\infty^p
\end{aligned}$$

Proof of (5).

$$\begin{aligned}
& \int_{B^n} |f(z)|^p |g(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1-p}} d\vartheta(z) \\
& C(\pi, m) \|g\|_\infty^p \leq \int_{B^n} (1-|\xi|^2)^{m+(p-1)(\gamma+1)} |Df(\xi)|^p \int_{B^n} \frac{\omega(1-|z|) d\vartheta(z) d\vartheta(\xi)}{|1-\langle z, \xi \rangle|^{m+n} (1-|z|^2)^{n+1-p+\gamma p/q}} \\
& \leq \int_{B^n} (1-|\xi|^2)^{m+(p-1)(\gamma+1)} \frac{|Df(\xi)|^p \omega(1-|\xi|)}{(1-|\xi|)^{n+1-p+\gamma p/q+m-1}} d\vartheta(\xi) \\
& = C(\pi, m) \|g\|_\infty^p \int_{B^n} \frac{|Df(\xi)|^p \omega(1-|\xi|)(1-|\xi|)^p d\vartheta(\xi)}{(1-|\xi|^2)^{n+1-p}} \leq C(\pi, m) \|g\|_\infty^p \|f\|_{B_p(\omega)}^p
\end{aligned}$$

Let $0 < p \leq 1$. Then by Lemma 4

$$|f(z)|^p \leq C(\pi, m) \int_{B^n} \frac{(1-|\xi|^2)^{mp+(n+1)p} |Df(\xi)|^p d\vartheta(\xi)}{|1-\langle m, \xi \rangle|^{(m+1)p} (1-|\xi|^2)^{n+1}}$$

Then for (4) we get

$$\begin{aligned}
& \int_{B^n} |f(z)|^p |Dg(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1-p}} d\vartheta(z) \\
& \leq C(\pi, m) \|g\|_\infty^p \int_{B^n} \frac{(1-|\xi|^2)^{mp+(n+1)p} |Df(\xi)|^p}{(1-|\xi|^2)^{n+1}} \int_{B^n} \frac{\omega(1-|z|) d\vartheta(z) d\vartheta(\xi)}{|1-\langle z, \xi \rangle|^{(m+1)p} (1-|z|^2)} \\
& \leq C(\pi, m) \|g\|_\infty^p \int_{B^n} \frac{(1-|\xi|^2)^{mp+(n+1)p} |Df(\xi)|^p \omega(1-|\xi|)}{(1-|\xi|^2)^{n+1} (1-|\xi|^2)^{(m+1)p}} d\vartheta(\xi) \\
& = C(\pi, m) \|g\|_\infty^p \int_{B^n} \frac{|Df(\xi)|^p \omega(1-|\xi|)(1-|\xi|^2)^{n(p+1)-p} d\vartheta(\xi)}{(1-|\xi|^2)^{n+1-p}} = C(\pi, m) \|g\|_\infty^p \|f\|_{B_p(\omega)}^p
\end{aligned}$$

Finally we obtain (5)

$$\begin{aligned}
& \int_{B^n} |f(z)|^p |g(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{n+1-p}} d\vartheta(z) \\
& \leq C(\pi, m) \|g\|_\infty^p \int_{B^n} \frac{(1-|\xi|^2)^{mp+(n+1)p} |Df(\xi)|^p}{(1-|\xi|^2)^{n+1}} \frac{\omega(1-|\xi|) d\vartheta(\xi)}{(1-|\xi|^2)^{(m+1)p-n-1-p}} \\
& \leq C(\pi, m) \|f\|_{B_p(\omega)}^p \|g\|_\infty^p
\end{aligned}$$

Summing ab, we get the proof of theorem. □

References

- [1] J. Arazy, S. Fisher, J. Peetre, Mobius invariant function spaces, *J. Reine Angew. Math.* **363**, (1985), 110-145
- [2] A. Djrbashian, F. Shamoyan, *Topics in the Theory of $A^p(\alpha)$ spaces.* (1988) Teubner Texte Math. Leipzig
- [3] A.V. Harutyunyan, W. Lusky, ω - weighted holomorphic Besov space on the unit ball in C^n , *Comment. Math. Univ. Carolin* 52,1 (2011), 37-56
- [4] A.V. Harutyunyan, W. Lusky, Holomorphic Bloch spaces on the unit ball in C^n . *Comment. Math. Univ. Carolin* 50,4(2009), 549-562
- [5] A.N. Karapetyants, F.D. Kodzoeva, Analytic weighed Besov spaces on the unit disc. *Proc.A.Razmadze Math.Inst.* **139**, (2005), 125-127.
- [6] M. Nowak, Bloch and Möbius invariant Besov spaces on the unit ball of C_n , *Complex Var.* **44**, (2001), 1-12.
- [7] W. Rudin, *Function theory in unit ball of C^n* Springer (1980)
- [8] S. Li, S. Stevic, Some characterizations of the Besov space and the α -Bloch space, *J. Math. Anal. Appl.* **346**, (2008), 262-273
- [9] S. Li, H. Wulan, Besov space in the unit ball, *Indian J. Math.* **48** (2) (2006), 177-186.
- [10] E. Seneta, *Functions of Regular Variation* [in Russian], Nauka, Moscow (1985)

- [11] K. Stroethoff, Besov type characterisations for the Bloch space, *Bull. Australian Math. Soc.* , **39**,(1989), 405-420
- [12] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball. Springer (2004)